BF Actions for the Husain-Kuchař Model.

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ABSTRACT

We show that the Husain-Kuchař model can be described in the framework of BF theories. This is a first step towards its quantization by standard perturbative QFT techniques or the spin-foam formalism introduced in the space-time description of General Relativity and other diff-invariant theories. The actions that we will consider are similar to the ones describing the BF-Yang-Mills model and some mass generating mechanisms for gauge fields. We will also discuss the role of diffeomorphisms in the new formulations that we propose.

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I Introduction

BF theories were introduced by Horowitz [1] and, independently, by Blau and Thompson [2] in the late eighties and early nineties. They are simple diff-invariant models that have been extensively studied as a testbed for quantization techniques in the absence of metric backgrounds. They have also been used as building blocks for physical theories such as gravity [3] and Yang-Mills [4]. From the point of view of their quantum treatment it is interesting to point out that they can be quantized both by using traditional Quantum Field Theory (QFT) techniques and methods specially tailored to deal with diff-invariant background-free actions such as spin networks and spin-foams [5].

The Husain-Kuchař model (HK) [6] is an interesting diff-invariant system with local degrees of freedom that actually mimics many of the features of 3+1 dimensional general relativity (GR). In fact, the solutions to the Einstein equations (with an internal $SO(3)$ group) can be seen as a subset of the solutions to HK. Its Hamiltonian description is very similar to the Ashtekar formulation for GR because both share the same phase space and most of the constraints –only the Hamiltonian constraint is missing in HK–. Its absence means that the model describes equivalence classes of metrics[3] in the spatial slices of a 3+1 foliation of space-time, but no time evolution. It is widely believed that a successful quantization of HK, and the implementation of efficient computational techniques to deal with it, would be important steps towards the quantization of full GR.

The standard action for the HK model [6] is very similar to the self-dual one [7] for GR because they essentially differ only in the internal gauge group ($SO(3,1)$ for GR and $SO(3)$ for HK[4]). This explains the similarities of their Hamiltonian descriptions. Some other action principles for this model have also appeared in the literature [8]-[10]. They provide different Hamiltonian formulations of the theory and solve some apparent shortcomings of the original formulation –such as the description of non-

[3]Under both diffeomorphisms and $SO(3)$ internal rotations.

[4]Actually the self-dual action in the Euclidean case can be obtained from the HK action by adding a term to it as shown in [9].

degenerate 4-metrics and the possibility to couple ordinary matter–. One of them (the two-connection formulation [8]) will be used in this paper as the starting point to obtain the HK model by coupling BF Lagrangians. The action that we give is very similar in form to the ones appearing in [4] for BF-Yang-Mills (BFYM) models, in [11] for a mass generating mechanism for gauge fields, or in [5] to discuss the problem of infrared divergencies in the spin-foam quantization of the BF theory.

There are several reasons that lead us to believe that the actions that we consider here are of interest. First of all they are structurally very simple because they consist only of several BF terms plus quadratic interactions. At variance with the known formulations, they have quadratic terms that may make it simpler to use perturbative QFT techniques. Also, we want to emphasize that we are not resorting to the usual (and rather trivial) trick of taking a BF model and impose restrictions on the 2-form field B with Lagrange multipliers because we want to preserve the quadratic character of the action as much as possible. Second, BF theories have also been quantized by using the so called spin-foams that provide a picture of quantum space-time geometry. As is shown in [5] one may need to add a "cosmological constant" term to remove apparent infrared divergencies that pop up in the computation of transition amplitudes[5]. This term is very similar to the ones that we introduce here so we hope that the strong similarity of our formulation with this one will provide a completely different way to deal with the HK model. Some of the issues that are not yet fully understood in the spin-foam framework, such as the relationship between the presence of local degrees of freedom and triangularization independence may be illuminated by using the actions that we give here.

An additional reason to consider different formulations for old models is the following. As it has been discussed by some authors [12], [13] the non-renormalizability of GR can be traced back to the non-invertibility of the quadratic part of the action after gauge fixing of its symmetries. In the case of the Hilbert-Palatini action there is, in fact, no quadratic term to build field propagators. Something similar

[5]The functional integrals in [5] are performed with the naive measure in field space without introducing the terms needed to take care of the second class constraints present in the theory.

happens for the previously known actions for the HK model because all of them lack a quadratic part. A possible way out of this is to consider the non-quadratic actions as interaction terms of theories modified by the addition of quadratic kinetic terms and try to extract some information from these modified models. For this approach to work it must be possible to add these kinetic terms in a consistent way, i.e. the fully interacting theory must be a consistent deformation (in the sense of [14]) of its quadratic part. As it turns out it is actually impossible to modify the theory by adding kinetic terms (both diff-invariant and in the presence of a metric background) to have a consistent formulation [15] where the Hilbert-Palatini or the HK actions appear as interactions. This is a consequence of the fact that these actions are built in terms of 1-form fields, so it is interesting to study whether new formulations in terms of different objects –such as 2-forms– will allow us to successfully incorporate consistent kinetic terms, or hopefully, define propagators.

A final issue that we will discuss is the role of diffeomorphisms in HK actions. We will show that, as in the 2+1 gravity case (a BF theory itself), one can actually write down non-diff invariant actions leading to the same field equations and gauge symmetries (on shell) as those of the standard formulations.

The paper is organized as follows. After this introduction we give, in section II, the BF description of the HK model –following a brief discussion of some of the other known formulations– and the general mechanism at work here and in other BF descriptions of known theories. Section III contains our conclusions and comments. We end the paper with an appendix where we discuss the detailed Dirac analysis for the actions that we present in the paper.

II BF Formulation of the Husain-Kuchař Model

BF theories in four dimensions are described by the action

$$S_{BF} = \int_{\mathcal{M}} d^4x\ \tilde{\eta}^{abcd} B_{abi} F^i_{cd}\,, \tag{1}$$

where $\tilde{\eta}^{abcd}$ is the four-dimensional Levi-Civita tensor density, F^i_{ab} is the curvature of a gauge connection 1-form A^i_a (taking values in the Lie algebra of a certain gauge group G) defined as $F^i_{ab} = 2\partial_{[a}A^i_{b]} + [A_a, A_b]^i$, and B_{abi} is a 2-form defined in the dual Lie algebra. Here and in the following we use tangent space indices a, b, c,… In this paper we will restrict ourselves to $G = SO(3)$; and use internal indices $i,\ j,\ k, ... = 1, 2, 3$. The totally antisymmetric Levi-Civita tensor will be denoted as ϵ_{ijk}. The differentiable manifold $\mathcal{M}$ is taken to have the topology $\mathcal{M} = \mathrm{I\!R} \times \Sigma$ with Σ a three-dimensional compact manifold without boundary.

The action for the Husain-Kuchař model is

$$S_{HK} = \int_{\mathcal{M}} d^4x\ \tilde{\eta}^{abcd}\epsilon_{ijk}e^i_a e^j_b F^k_{cd}. \tag{2}$$

where $F^i_{ab} = 2\partial_{[a}A^i_{b]} + \epsilon^{ijk}A_{aj}A_{bk}$ is the curvature of a $SO(3)$ connection A^i_a and e^i_a are three 1-forms (notice that they are not proper tetrads because there are only three of them). As shown in [8] it can be rewritten (modulo surface terms) as

$$S'_{HK} = \int_{\mathcal{M}} d^4x\ \tilde{\eta}^{abcd}\overset{1}{F}_{abi}\overset{2}{F}{}^i_{cd}\,, \tag{3}$$

where $\overset{1}{F}_{abi}$ and $\overset{2}{F}_{abi}$ are the curvatures of two $SO(3)$ connections $\overset{1}{F}{}^i_{ab} = 2\partial_{[a}\overset{1}{A}{}^i_{b]} + \epsilon^{ijk}\overset{1}{A}_{aj}\overset{1}{A}_{bk}$ and $\overset{2}{F}{}^i_{ab} = 2\partial_{[a}\overset{2}{A}{}^i_{b]} + \epsilon^{ijk}\overset{2}{A}_{aj}\overset{2}{A}_{bk}$. This can be easily seen by defining $e_{ai} = \overset{2}{A}_{ai} - \overset{1}{A}_{ai}$ and using the fact that

$$\overset{2}{F}{}^i_{ab} = \overset{1}{F}{}^i_{ab} + 2\overset{1}{\nabla}_{[a}e^i_{b]} + \epsilon^{ijk}e_{aj}e_{bk} \tag{4}$$

(the covariant derivative $\overset{1}{\nabla}_a$ acts on internal $SO(3)$ indices as $\overset{1}{\nabla}_a\lambda_i = \partial_a\lambda_i + \epsilon_i{}^{jk}\overset{1}{A}_{aj}\lambda_k$ and $\overset{2}{\nabla}_a$ is defined in an analogous way). By using (4) we can write the action (2) as

$$S'_{HK} = \int_{\mathcal{M}} d^4x\ \tilde{\eta}^{abcd}\left[\overset{1}{F}_{abi}\overset{1}{F}{}^i_{cd} + 2\overset{1}{F}{}^i_{ab}\overset{1}{\nabla}_c e_{di} + \epsilon^{ijk}e_{ai}e_{bj}\overset{1}{F}_{cdk}\right]; \tag{5}$$

dropping the first term due to its topological character and the second after integrating by parts and using the Bianchi identities we end up with the action (2). Notice that both $\overset{1}{F}{}^i_{ab}$ and $\overset{2}{F}{}^i_{ab}$ are defined in the Lie algebra of $SO(3)$ so we must use the invariant metric δ^{ij} to build the action. The field equations coming from (3) have the beautifully symmetric form

$$\left\{\begin{array}{l} \overset{1}{\nabla}_{[a}\overset{2}{F}{}^i_{bc]} = 0 \\ \\ \overset{2}{\nabla}_{[a}\overset{1}{F}{}^i_{bc]} = 0\,. \end{array}\right. \tag{6}$$

Let us consider now the following action[6]

$$S_{BFHK}=\int_{\mathcal{M}} d^4x\,\tilde{\eta}^{abcd}\left[\overset{1}{B}_{abi}\overset{1}{F}{}^i_{cd}+\overset{2}{B}_{abi}\overset{2}{F}{}^i_{cd}+\overset{1}{B}_{abi}\overset{2}{B}{}^i_{cd}\right], \tag{7}$$

obtained by coupling two BF theories via a $\tilde{\eta}^{abcd}\overset{1}{B}_{abi}\overset{2}{B}{}^i_{cd}$ interaction term. The field equations coming from (7) are now

$$\left\{\begin{array}{l}\overset{1}{F}{}^i_{ab}+\overset{2}{B}{}^i_{ab}=0\\[2ex] \overset{2}{F}{}^i_{ab}+\overset{1}{B}{}^i_{ab}=0\end{array}\right.\qquad\left\{\begin{array}{l}\overset{1}{\nabla}_{[a}\overset{1}{B}{}^i_{bc]}=0\\[2ex] \overset{2}{\nabla}_{[a}\overset{2}{B}{}^i_{bc]}=0\,.\end{array}\right. \tag{8}$$

The equivalence between these equations and (6) is obvious. The new action S_{BFHK} is both $SO(3)$ and diff-invariant. To understand how we get local degrees of freedom from the two topological BF models it suffices to realize that an action obtained by adding two different BF terms has two independent sets of symmetries that are partially broken by the $\tilde{\eta}^{abcd}\overset{1}{B}_{abi}\overset{2}{B}{}^i_{cd}$ term; specifically, without this coupling the action would be invariant under two independent sets of $SO(3)$ rotations and diffeomorphisms. The reduced gauge symmetry accounts for the new local degrees of freedom described by (7). By using the terminology introduced by Smolin in [3] the action (7) describes the HK model as a constrained topological field theory.

There is an evident, one to one, correspondence between the solution spaces to equations (6) and (8) so, classically, both actions (3) and (7) describe the same theory. The equivalence of their quantum formulations is, however, less obvious. As S_{BFHK} is quadratic in $\overset{1}{B}{}^i_{ab}$ and $\overset{2}{B}{}^i_{ab}$ one would be tempted to say that the effect of performing the functional integration in these fields would be equivalent to substituting the field equations for them in the action, in which case we would end up with the 2-connection action (3) and a functional integral measure involving only the connections. However this is not correct because the functional measure is not the naive one but must be modified [16] to take into account the existence of second class constraints in the Hamiltonian formulation derived from (7). An intuitively clear way to understand

[6]Here we are following a suggestion due to Giorgio Immirzi to couple two Yang-Mills theories by using their BFYM formulations.

this fact is to realize that, as a consequence of the Bianchi identities, only those components of $\overset{1}{B}{}^i_{ab}$ and $\overset{2}{B}{}^i_{ab}$ that cannot be written as covariant derivatives (in terms of the connections $\overset{1}{A}{}^i_{ab}$ and $\overset{2}{A}{}^i_{ab}$) couple to the curvatures $\overset{1}{F}{}^i_{ab}$ and $\overset{2}{F}{}^i_{ab}$ respectively. One must, in fact, remove them form the path integral in order to avoid problems such as the violation of the Ward identities that one encounters in BFYM models [4]. After modifying the measure to circumvent this difficulty the integration in $\overset{1}{B}{}^i_{ab}$ and $\overset{2}{B}{}^i_{ab}$ is no longer trivial. The Dirac analysis of (7) is rather involved because one expects to find many second class constraints. Most of them appear as consistency conditions for the solvability of the Lagrange multiplier equations that one must introduce in the total Hamiltonian. A way to partially alleviate this problem is to use Stückelberg's procedure [17] in (7) by introducing auxiliary fields $\overset{1}{\eta}{}^i_a$ and $\overset{2}{\eta}{}^i_a$ in the action and trading some second class constraints for first class constraints. So we will also consider

$$S'_{BFHK} = \int_{\mathcal{M}} d^4x\, \tilde{\eta}^{abcd} \left[\overset{1}{B}_{abi} \overset{1}{F}{}^i_{cd} + \overset{2}{B}_{abi} \overset{2}{F}{}^i_{cd} + (\overset{1}{B}_{abi} - \overset{1}{\nabla}_a \overset{1}{\eta}{}^i_b)(\overset{2}{B}{}^i_{cd} - \overset{2}{\nabla}_c \overset{2}{\eta}{}^i_d) \right] . \tag{9}$$

The field equations derived from (9) are

$$\begin{aligned}
&\left\{ \begin{array}{l} \overset{1}{F}{}^i_{ab} + \overset{2}{B}{}^i_{ab} - \overset{2}{\nabla}_{[a} \overset{2}{\eta}{}^i_{b]} = 0 \\[2ex] \overset{2}{F}{}^i_{ab} + \overset{1}{B}{}^i_{ab} - \overset{1}{\nabla}_{[a} \overset{1}{\eta}{}^i_{b]} = 0 \,, \end{array} \right. \\
&\left\{ \begin{array}{l} 2\overset{1}{\nabla}_{[a} \overset{1}{B}{}^i_{bc]} - \epsilon^{ijk} \overset{1}{\eta}{}^j_{[a} \left(\overset{2}{B}{}^k_{bc]} - \overset{2}{\nabla}_b \overset{2}{\eta}{}^k_{c]} \right) = 0 \\[2ex] 2\overset{2}{\nabla}_{[a} \overset{2}{B}{}^i_{bc]} - \epsilon^{ijk} \overset{2}{\eta}{}^j_{[a} \left(\overset{1}{B}{}^k_{bc]} - \overset{1}{\nabla}_b \overset{1}{\eta}{}^k_{c]} \right) = 0 \,, \end{array} \right.
\end{aligned} \tag{10}$$

$$\left\{ \begin{array}{l} \overset{1}{\nabla}_{[a} \left(\overset{2}{B}{}^i_{bc]} - \overset{2}{\nabla}_b \overset{2}{\eta}{}^i_{c]} \right) = 0 \\[2ex] \overset{2}{\nabla}_{[a} \left(\overset{1}{B}{}^i_{bc]} - \overset{1}{\nabla}_b \overset{1}{\eta}{}^i_{c]} \right) = 0 \,. \end{array} \right.$$

From the first pair we find

$$\left\{ \begin{array}{l} \overset{1}{B}{}^i_{ab} = -\overset{2}{F}{}^i_{ab} + \overset{1}{\nabla}_{[a} \overset{1}{\eta}{}^i_{b]} \\[2ex] \overset{2}{B}{}^i_{ab} = -\overset{1}{F}{}^i_{ab} + \overset{2}{\nabla}_{[a} \overset{2}{\eta}{}^i_{b]} \,, \end{array} \right. \tag{11}$$

so that the remaining equations give (6). After solving them we can obtain $\overset{1}{B}{}^{i}_{ab}$ and $\overset{2}{B}{}^{i}_{ab}$ (modulo the arbitrariness in $\overset{1}{\eta}{}^{i}_{a}$ and $\overset{2}{\eta}{}^{i}_{a}$) from (11). Notice that the third pair of equations in (10) is just a consequence of the first and the Bianchi identities.

In addition to the $SO(3)$ and the diff-invariance of (7) this action is invariant under the transformations

$$\left\{\begin{array}{l} \delta \overset{1}{A}{}^{i}_{a} = 0 \\ \delta \overset{2}{A}{}^{i}_{a} = 0 \\ \delta \overset{1}{B}{}^{i}_{ab} = \overset{1}{\nabla}_{[a} \overset{1}{\varphi}{}^{i}_{b]} \\ \delta \overset{2}{B}{}^{i}_{ab} = 0 \\ \delta \overset{1}{\eta}{}^{i}_{a} = \overset{1}{\varphi}{}^{i}_{a} \\ \delta \overset{2}{\eta}{}^{i}_{a} = 0 \end{array}\right. \qquad \left\{\begin{array}{l} \delta \overset{1}{A}{}^{i}_{a} = 0 \\ \delta \overset{2}{A}{}^{i}_{a} = 0 \\ \delta \overset{1}{B}{}^{i}_{ab} = 0 \\ \delta \overset{2}{B}{}^{i}_{ab} = \overset{2}{\nabla}_{[a} \overset{2}{\varphi}{}^{i}_{b]} \\ \delta \overset{1}{\eta}{}^{i}_{a} = 0 \\ \delta \overset{2}{\eta}{}^{i}_{a} = \overset{2}{\varphi}{}^{i}_{a} \end{array}\right. \tag{12}$$

where $\overset{1}{\varphi}{}^{i}_{a}$ and $\overset{2}{\varphi}{}^{i}_{a}$ are arbitrary gauge parameters. A possible way to ensure that these are the only symmetries of the proposed action (and a first step towards the quantization of the model) is to study its Hamiltonian formulation. This is done in the appendix. One can see that the model is, in fact, equivalent to the 2-connection formulation after performing a suitable gauge fixing of the symmetries introduced by the Stückelberg procedure. Here we follow a different path to prove the equivalence of S_{BFHK} and S'_{BFHK} with (3) *at the quantum level* by using the covariant symplectic techniques introduced in [18].

To this end we must look at the symplectic form in the space of solutions to the field equations [18]. This is obtained in a two-step process; first we write a closed (but degenerate at this stage) 2-form in the space of fields. For a Lagrangian of the type $L(\varpi, d\varpi)$ depending on a s-form field ϖ and its exterior derivative it is given by

$$\Omega = \int_{\Sigma} \mathbb{d}\varpi \mathbin{\wedge\!\!\!\wedge} \mathbb{d}\frac{\partial L}{\partial d\varpi}\,, \tag{13}$$

where we define $\frac{\partial L}{\partial d\varpi}$ according to

$$L(d\varpi + d\epsilon) = L(d\varpi) + d\epsilon \wedge \frac{\partial L}{\partial d\varpi} + \text{higher order}\,.$$

Here we use the differential form notation to make a clear distinction between spacetime and phase space objects. We must distinguish between the ordinary exterior differential in the spacetime manifold $\mathcal{M}$ (d) and the exterior differential in the field space($\mathrm{d\!I}$). In the same way we must make a distinction between the wedge product in both cases ($\wedge$ and $\mathbin{\wedge\!\!\!\wedge}$ respectively). In the following we will avoid this by explicitly writing tangent space indices. Notice that d is defined in $\mathcal{M}$ but the integral in (13) is three-dimensional.

We must now pull-back Ω to the space of solutions to the field equations. Whenever this can be explicitly done (in those few cases where the solutions to the field equations can be completely parametrized [15]) one gets a closed and non-degenerate 2-form in the space of solutions to the field equations. The non-degeneracy implies that no gauge freedom remains[7] and the objects that appear in the symplectic form parametrize physical degrees of freedom. For the action (7) we have[8]

$$\Omega = 2\int_\Sigma d^3x\,\tilde{\eta}^{abc}\left[\mathrm{d\!I}\,\overset{1}{A}{}^i_a(x)\mathbin{\wedge\!\!\!\wedge}\mathrm{d\!I}\,\overset{1}{B}_{bci}(x)+\mathrm{d\!I}\,\overset{2}{A}{}^i_a(x)\mathbin{\wedge\!\!\!\wedge}\mathrm{d\!I}\,\overset{2}{B}_{bci}(x)\right]\,,$$

where $\tilde{\eta}^{abc}$ is the three dimensional Levi-Civita tensor density in the x-coordinate patch and $\overset{1}{A}{}^i_a$, $\overset{2}{A}{}^i_a$, $\overset{1}{B}{}^i_{ab}$, and $\overset{2}{B}{}^i_{ab}$ are the pull-backs of the corresponding four dimensional objects onto Σ. After partially pulling back to the solution space by using the field equations expressing $\overset{1}{B}$ and $\overset{2}{B}$ in terms of the curvatures we get[9]

$$\Omega = -4\int_\Sigma d^3x\,\tilde{\eta}^{abc}\left\{\mathrm{d\!I}\,\overset{1}{A}{}^i_a(x)\mathbin{\wedge\!\!\!\wedge}[\overset{2}{\nabla}_b-\overset{1}{\nabla}_b]\mathrm{d\!I}\,\overset{2}{A}_{ci}(x)\right\}. \tag{14}$$

This is not yet the symplectic structure in the solution space because we still have to consider the remaining field equations but it coincides with the one derived from (3). This not only proves the equivalence of the classical theories but also of their quantum formulations because their reduced phase spaces coincide and have the same symplectic structure. For the action (9) we obtain in a similar fashion

$$\Omega=\int_\Sigma d^3x\;\tilde{\eta}^{abc}\left\{2\mathrm{d\!I}\,\overset{1}{A}{}^i_a(x)\mathbin{\wedge\!\!\!\wedge}\mathrm{d\!I}\,\overset{1}{B}_{bci}(x)+2\mathrm{d\!I}\,\overset{2}{A}{}^i_a(x)\mathbin{\wedge\!\!\!\wedge}\mathrm{d\!I}\,\overset{2}{B}_{bci}(x)-\right. \tag{15}$$

[7]This is equivalent to the reduced phase space of the usual Hamiltonian formulation.

[8]Notice that the $A(x)$ and $B(x)$ fields are time dependent; here d^3x denotes the measure in Σ.

[9]This coincides with the result in [8]. This example illustrates the efficiency of covariant symplectic methods as compared to the more traditional approach of Dirac.

$$-\mathrm{d}\!\!\mathrm{I}\overset{1}{\eta}{}^{i}_{a}(x)\wedge\!\!\!\wedge\mathrm{d}\!\!\mathrm{I}[\overset{2}{B}_{bci}(x)-\overset{2}{\nabla}_b\overset{2}{\eta}{}_{ci}(x)]-\mathrm{d}\!\!\mathrm{I}\overset{2}{\eta}{}^{i}_{a}(x)\wedge\!\!\!\wedge\mathrm{d}\!\!\mathrm{I}[\overset{1}{B}_{bci}(x)-\nabla_b\overset{1}{\eta}{}_{ci}(x)]\Big\}$$

and, as before, after partially pulling back to the solution space by using the field equations expressing $\overset{1}{B}$ and $\overset{2}{B}$ in terms of curvatures we get

$$\Omega=\int_\Sigma d^3x\tilde{\eta}^{abc}\Big\{\mathrm{d}\!\!\mathrm{I}\overset{1}{\eta}{}^{i}_{a}(x)\wedge\!\!\!\wedge\mathrm{d}\!\!\mathrm{I}\overset{1}{F}_{bci}(x)+\mathrm{d}\!\!\mathrm{I}\overset{2}{\eta}{}^{i}_{a}(x)\wedge\!\!\!\wedge\mathrm{d}\!\!\mathrm{I}\overset{2}{F}_{bci}(x)+ \tag{16}$$
$$+2\mathrm{d}\!\!\mathrm{I}\overset{1}{A}{}^{i}_{a}(x)\wedge\!\!\!\wedge\mathrm{d}\!\!\mathrm{I}[\nabla_b\overset{1}{\eta}{}_{ci}(x)-\overset{2}{F}_{bci}(x)]+2\mathrm{d}\!\!\mathrm{I}\overset{2}{A}{}^{i}_{a}(x)\wedge\!\!\!\wedge\mathrm{d}\!\!\mathrm{I}[\overset{2}{\nabla}_b\overset{2}{\eta}{}_{ci}(x)-\overset{1}{F}_{bci}(x)]\Big\}$$

which, after a little algebra gives precisely (14) and, hence, we also prove the classical and quantum equivalence of (9) with the actions considered before.

It is interesting to compare this situation with some other similar models. Let us consider first the BFYM theory that is known to be amenable to perturbative treatment [4]. The BFYM action can be written as

$$S_{BFYM}[A,B]=S_{YM}[A]-\int_{\mathcal{M}}d^4x\;(B^i_{ab}+\frac{1}{g}F^i_{ab})(B_{cdi}+\frac{1}{g}F_{cdi})\eta^{ac}\eta^{bd}\,,$$

where $S_{YM}[A]$ is the usual Yang-Mills action for A with a coupling constant g and η^{ab} is a background Minkowski metric. The term $(B+F/g)^2$ vanishes on shell thus making the BFYM and the YM models equivalent[10]. This term yields a propagator for B. Moreover S_{YM} has, after a suitable gauge fixing, a well defined propagator for the Yang-Mills field A.

In the BFHK case we can also write

$$S_{BFHK}[A_1,A_2,B]=S'_{HK}[A_1,A_2]-\int_{\mathcal{M}}d^4x\tilde{\eta}^{abcd}(\overset{1}{B}{}^{i}_{ab}+\overset{1}{F}{}^{i}_{ab})(\overset{1}{B}_{cdi}+\overset{1}{F}_{cdi})\,, \tag{17}$$

but now S'_{HK} does not have a free part and the quadratic form $(\overset{1}{B}+\overset{1}{F})(\overset{2}{B}+\overset{2}{F})$ is always degenerate in the A, B fields, even after gauge fixing. This prevents us from using standard QFT techniques because we do not have a suitable propagator even though we have a quadratic piece in the action. However (17) suggests a possible way out. The main problem that we face is related to the diff-invariance of the quadratic terms derived from S_{BFHK}. Looking at the mechanism that is working in (17), one can

[10]This argument, based on the resolution of algebraic field equations back in the action, relies on the following fact: S_{BFYM} action generates a covariant phase space that is in one-one correspondence with the Yang-Mills one. Moreover, it induces a symplectic two form over the space of A, B fields that coincides, when restricted to the solution space, with the symplectic Yang-Mills form.

devise a straightforward generalization where the diff-invariance is broken through the use of a metric background g, so let us consider, for example, the action

$$S'' = S'_{HK} - \int_{\mathcal{M}} (\overset{1}{B}{}^i_{ab} + * \overset{1}{F}{}^i_{ab})(* \overset{2}{B}_{cdi} + \overset{2}{F}_{cdi}) \eta^{ac} \eta^{bd} \,. \tag{18}$$

where $*F^i_{ab}$ is the Hodge dual F^i_{ab}. Once again, the equivalence with the HK model is due to the vanishing of the B-term of the action in the covariant phase space. Clearly, although S'' is not diff-invariant, the diff-invariance is recovered on shell. This formulation has the same perturbative objections that S_{BFHK}. However, the presence of a metric background opens up the possibility of adding to the action S'' kinetic terms for A and B. Then S'' might be considered as an interaction term of an action with well-defined propagators. Of course this would be possible only if these (local) kinetic terms can be incorporated in a consistent way, both from the classical and quantum point of view.

In addition to the BFYM models some other theories of this type have been studied in the literature. An interesting example is the mass generating mechanism for gauge fields proposed by Lahiri in [11] and described by the action

$$\int_{\mathcal{M}} d^4x \, Tr \left[-\frac{1}{6} H_{abc} H^{abc} - \frac{1}{4} F_{ab} F^{ab} + \frac{m}{2} \tilde{\eta}^{abcd} B_{ab} F_{cd} \right] , \tag{19}$$

with $H_{abc} = 3\nabla_{[a} B_{bc]}$ and m is a mass parameter. The essential difference between our model and Lahiri's is related to how the B field appears in the action. In [11], after disregarding total derivatives, one can write for the linear theory

$$\int_{\mathcal{M}} d^4x Tr \left\{ -\partial_{[a} A_{b]} \partial^a A^b - \frac{m^2}{4} A_a A^a - \frac{3}{2} [\partial_{[a} B_{bc]} - \frac{m}{6} \epsilon_{abcd} A^d][\partial^a B^{bc} - \frac{m}{6} \epsilon^{abce} A_e] \right\} \tag{20}$$

but now the $(dB - m * A)^2$ term does not vanish on shell but it is proportional to the gradient of the gauge parameter for the A field. Another property of [11], that makes his model closer to S'', is the impossibility of using standard perturbative treatments [19]. This is so because the free Lagrangian (20) has two independent gauge symmetries

$$\begin{array}{ll} \delta_1 A^i_a = \partial_a \Lambda^i & \qquad \delta_2 A^i_a = 0 \\ \delta_1 B^i_{ab} = 0 & \qquad \delta_2 B^i_{ab} = \partial_{[a} \Lambda^i_{b]} \,, \end{array}$$

for arbitrary 0-forms Λ^i and 1-forms Λ^i_a while the full model has only a $SO(3)$ symmetry. This problem is exactly the same that afflicts the Palatini gravitational theories in four space-time dimensions [15]. The open question is, again, the existence of consistent theories, in the perturbative sense, for interacting 1-form and 2-form fields implementing this type mechanism. As a systematic study of quadratic actions for 1-form and 2-form fields has not been carried out to date we hope that suitable theories can be found.

III Conclusions and Comments

As we have shown in the paper the Husain-Kuchař model can be obtained by coupling BF theories. The mechanism at work in the actions (7) and (9) is different from the standard procedure of writing a BF action and imposing constraints on the 2-form field with Lagrange multipliers. In the examples that we consider in this paper the local degrees of freedom of the model (three per space point) appear due to the breaking of some of the symmetries of the BF terms induced by the introduction of a "cosmologial constant" term coupling the 2-form fields $\overset{1}{B}$ and $\overset{2}{B}$. The main reason why we do this is that we want to have a well defined quadratic part in the action in order to define propagators (after a suitable gauge fixing of the symmetries) and treat the model by using standard perturbative QFT techniques.

Of course, even in the presence of a quadratic part, the theory may not have suitable propagators. This happens if the symmetries of the quadratic part do not match the symetries of the full action (i.e. the full action is not a consistent deformation of its quadratic part). In the present case we do not get a perturbatively quantizable action because, as shown in [15], its kinetic term is diff-invariant so it describes no local degrees of freedom and hence it has more symmetries than the full action. Anyway, it may be possible to modify these actions by the addition of local kinetic terms that may render the quadratic part invertible in such a way that the full action is a consistent deformation of it. If this can be achieved it may be possible to effectively recover the HK model in some limit, or at least extract useful information from its behavior as an interaction term. One of the points that we emphasize in the paper is precisely the fact that writing the action in terms of new types of fields it may be possible to have kinetic terms not available in the known formulations.

We have shown that one can actually derive the HK models from non diff-invariant actions. This is similar to GR because, as is well known [20], it is possible to derive the Einstein equations from a non-diff-invariant action, quadratic in the Christoffel connections, obtained by integrating by parts the second order terms in the expression for the scalar curvature. In this case the loss of symmetry is compensated by the appearance of many second class constraints that reduce the number of physical degrees of freedom to the two physical ones. Diff-invariance is kept only on-shell. Again, the availability of a background structure may permit the introduction of kinetic terms that could not be written without its help.

We want to point out that covariant symplectic techniques are of great help to study the physical content of field theories without having to rely on the more traditional Dirac analysis. We have shown that it is very easy to prove the classical and quantum equivalence of the actions that we have discussed here. This requires some comments. In principle, two non-trivially different actions[11] may lead to the very same field equations. This does not mean, however, that they are equivalent from the quantum point of view. To prove their quantum equivalence the actions must endow their solution spaces with the same symplectic structure. This does indeed happen in this case as shown above but this should not happen in general.

In a precise sense it is not necessary to go through the painful procedure of using perturbative techniques as in [4] to show that the BFYM action is equivalent, from the quantum point of view to the usual Yang-Mills one. Perturbation theory, if properly done (and disregarding difficult issues of convergence), *reduces to canonical quantization*. If one can argue, as we did before, that the solution spaces of the usual Yang-Mills formulation and BFYM coincide and are endowed by the action with the same sympletic structure their quantum formulations must coincide (which is the

[11]By non-trivial we mean that they not differ in total derivatives or cannot be transformed into one another by simple redefinitions of the objects in terms of which they are described.

result that one finds out, at the end of the day).

IV Appendix: Dirac Analysis of the Coupled BF Action for the HK Model

We give a detailed Dirac analysis for the action (9) to show that we arrive exactly at the same conclusions that we got by using covariant symplectic techniques. We start by introducing a foliation of the space-time manifold $\mathcal{M} = \mathbb{R} \times \Sigma$ (where Σ denotes a compact 3-manifold without a boundary) defined by a scalar field t and a congruence of curves nowhere tangent to the $t = \textit{constant}$ hypersurfaces parametrized by t. In this way, if t is used as a "time coordinate" we have $\partial_t = \mathcal{L}_{\mathbf{t}}$, i.e. time derivatives are given by the Lie derivatives along the tangent vectors $\mathbf{t}$ to the curves of the congruence parametrized by t. In the following we make no distinction between three dimensional and four dimensional tangent indices as it will be clear from the context if we are considering 3-dim or 4-dim objects. By using the identity $\tilde{\eta}^{abcd} = 4t^{[a}\tilde{\eta}^{bcd]}$, where $\tilde{\eta}^{bcd}$ is the Levi-Civita tensor density in three dimensions, and defining

$$
\begin{cases} t^a \overset{1}{B}{}_{ab}{}^i \equiv \overset{1}{\beta}{}^i_b \\ t^a \overset{2}{B}{}_{ab}{}^i \equiv \overset{2}{\beta}{}^i_b \end{cases}
\quad
\begin{cases} \tilde{\eta}^{abc} \overset{1}{F}{}_{bci} \equiv \overset{1}{\tilde{F}}{}^a_i \\ \tilde{\eta}^{abc} \overset{2}{F}{}_{bci} \equiv \overset{2}{\tilde{F}}{}^a_i \end{cases}
\quad
\begin{cases} t^a \overset{1}{\eta}{}^i_a \equiv \overset{1}{\eta}{}^i_0 \\ t^a \overset{2}{\eta}{}^i_a \equiv \overset{2}{\eta}{}^i_0 \end{cases}
$$
$$
\begin{cases} \tilde{\eta}^{abc} \overset{1}{B}{}_{bc}{}^i \equiv \overset{1}{\tilde{B}}{}^a_i \\ \tilde{\eta}^{abc} \overset{2}{B}{}_{bc}{}^i \equiv \overset{2}{\tilde{B}}{}^a_i \end{cases}
\quad
\begin{cases} t^a \overset{1}{A}{}^i_a \equiv \overset{1}{A}{}^i_0 \\ t^a \overset{2}{A}{}^i_a \equiv \overset{2}{A}{}^i_0 \end{cases}
\tag{21}
$$

we find

$$
\begin{aligned}
&\int dt \Big\{ \overset{1}{\tilde{B}}{}^a_i \mathcal{L}_{\mathbf{t}} \overset{1}{A}{}^i_a + \overset{2}{\tilde{B}}{}^a_i \mathcal{L}_{\mathbf{t}} \overset{2}{A}{}^i_a + \frac{1}{2}\left(\tilde{\eta}^{abc} \overset{2}{\nabla}_b \overset{2}{\eta}{}^i_c - \overset{2}{\tilde{B}}{}^a_i \right) \mathcal{L}_{\mathbf{t}} \overset{1}{\eta}{}^i_a + \frac{1}{2}\left(\tilde{\eta}^{abc} \overset{1}{\nabla}_b \overset{1}{\eta}{}^i_c - \overset{1}{\tilde{B}}{}^a_i \right) \mathcal{L}_{\mathbf{t}} \overset{2}{\eta}{}^i_a + \\
&+ \overset{1}{\beta}{}^i_a \left(\overset{1}{\tilde{F}}{}^a_i + \overset{2}{\tilde{B}}{}^a_i - \tilde{\eta}^{abc} \overset{2}{\nabla}_b \overset{2}{\eta}{}^i_c \right) + \overset{2}{\beta}{}^i_a \left(\overset{2}{\tilde{F}}{}^a_i + \overset{1}{\tilde{B}}{}^a_i - \tilde{\eta}^{abc} \overset{1}{\nabla}_b \overset{1}{\eta}{}^i_c \right) + \\
&+ \overset{1}{A}{}^i_0 \left[\overset{1}{\nabla}_a \overset{1}{\tilde{B}}{}^a_i + \frac{1}{2}\epsilon^{ijk} \left(\overset{2}{\tilde{B}}{}^a_j - \tilde{\eta}^{abc} \overset{2}{\nabla}_b \overset{2}{\eta}{}^j_c \right) \overset{1}{\eta}{}_{ak} \right] + \frac{1}{2} \overset{1}{\eta}{}^i_0 \overset{1}{\nabla}_a \left(\tilde{\eta}^{abc} \overset{2}{\nabla}_b \overset{2}{\eta}{}^i_c - \overset{2}{\tilde{B}}{}^a_i \right) +
\end{aligned}
\tag{22}
$$

$$+\overset{2}{A}{}^i_0\left[\overset{2}{\nabla}_a\overset{2}{\tilde{B}}{}^a_i+\frac{1}{2}\epsilon^{ijk}\left(\overset{1}{\tilde{B}}{}^a_j-\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}^j_c\right)\overset{2}{\eta}{}_{ak}\right]+\frac{1}{2}\overset{2}{\eta}{}^i_0\overset{2}{\nabla}_a\left(\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}^i_c-\overset{1}{\tilde{B}}{}^a_i\right)\bigg\}\,.$$

The momenta canonically conjugate to the variables[12] $\overset{1}{A}{}^i_a$, $\overset{2}{A}{}^i_a$, $\overset{1}{\tilde{B}}{}^a_i$, $\overset{2}{\tilde{B}}{}^a_i$, $\overset{1}{A}{}^i_0$, $\overset{2}{A}{}^i_0$, $\overset{1}{\beta}{}^a_i$, $\overset{2}{\beta}{}^a_i$, $\overset{1}{\eta}{}^a_i$, $\overset{2}{\eta}{}^a_i$, $\overset{1}{\eta}{}^i_0$, and $\overset{2}{\eta}{}^i_0$ will be denoted as $\overset{1}{\tilde{\pi}}{}^a_i$, $\overset{2}{\tilde{\pi}}{}^a_i$, $\overset{1}{\sigma}{}^i_a$, $\overset{2}{\sigma}{}^i_a$, $\overset{1}{\tilde{\pi}}{}_i$, $\overset{2}{\tilde{\pi}}{}_i$, $\overset{1}{\tilde{\nu}}{}^a_i$, $\overset{2}{\tilde{\nu}}{}^a_i$, $\overset{1}{\tilde{\psi}}{}^a_i$, $\overset{2}{\tilde{\psi}}{}^a_i$, $\overset{1}{\tilde{\psi}}{}_i$, and $\overset{2}{\tilde{\psi}}{}_i$. They satisfy the following primary constraints

$$\begin{cases}\overset{1}{\tilde{\pi}}{}^a_i-\overset{1}{\tilde{B}}{}^a_i=0\\ \overset{2}{\tilde{\pi}}{}^a_i-\overset{2}{\tilde{B}}{}^a_i=0\end{cases}\quad\begin{cases}\overset{1}{\tilde{\pi}}{}_i=0\\ \overset{2}{\tilde{\pi}}{}_i=0\end{cases}\quad\begin{cases}\overset{1}{\sigma}{}^i_a=0\\ \overset{2}{\sigma}{}^i_a=0\end{cases}$$
$$\begin{cases}\overset{1}{\tilde{\nu}}{}^a_i=0\\ \overset{2}{\tilde{\nu}}{}^a_i=0\end{cases}\quad\begin{cases}\overset{1}{\tilde{\psi}}{}_i=0\\ \overset{2}{\tilde{\psi}}{}_i=0\end{cases}\quad\begin{cases}\overset{1}{\tilde{\psi}}{}^a_i-\frac{1}{2}\left(\tilde{\eta}^{abc}\overset{2}{\nabla}_b\overset{2}{\eta}{}^i_c-\overset{2}{\beta}{}^a_i\right)=0\\ \overset{2}{\tilde{\psi}}{}^a_i-\frac{1}{2}\left(\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}^i_c-\overset{1}{\beta}{}^a_i\right)=0\,.\end{cases}\tag{23}$$

The Hamiltonian is

$$\begin{aligned}H=-\int_\Sigma d^3y\,\Big\{&\overset{1}{\beta}{}^i_a\left[\overset{1}{\tilde{F}}{}^a_i+\overset{2}{\tilde{B}}{}^a_i-\tilde{\eta}^{abc}\overset{2}{\nabla}_b\overset{2}{\eta}{}_{ci}\right]+\overset{2}{\beta}{}^i_a\left[\overset{2}{\tilde{F}}{}^a_i+\overset{1}{\tilde{B}}{}^a_i-\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}_{ci}\right]+\\ &+\overset{1}{A}{}^i_0\left[\overset{1}{\nabla}_a\overset{1}{\tilde{B}}{}^a_i+\frac{1}{2}\left(\overset{2}{\tilde{B}}{}^a_j-\tilde{\eta}^{abc}\overset{2}{\nabla}_b\overset{2}{\eta}{}_{cj}\right)\overset{1}{\eta}{}_{ak}\right]+\\ &\qquad+\overset{2}{A}{}^i_0\left[\overset{2}{\nabla}_a\overset{2}{\tilde{B}}{}^a_i+\frac{1}{2}\epsilon_{ijk}\left(\overset{1}{\tilde{B}}{}^a_j-\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}_{cj}\right)\overset{2}{\eta}{}_{ak}\right]+\\ &+\frac{1}{2}\overset{1}{\eta}{}^i_0\overset{1}{\nabla}_a\left[\tilde{\eta}^{abc}\overset{2}{\nabla}_b\overset{2}{\eta}{}_{ci}-\overset{2}{\tilde{B}}{}^a_i\right]+\frac{1}{2}\epsilon_{ijk}\overset{2}{\eta}{}^i_0\overset{2}{\nabla}_a\left[\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}_{ci}-\overset{1}{\tilde{B}}{}^a_i\right]\Big\}\end{aligned}\tag{24}$$

and the total Hamiltonian

$$\begin{aligned}H_T=H+\int_\Sigma d^3y\Big\{&\overset{1}{l}{}^i_a(\overset{1}{\tilde{\pi}}{}^a_i-\overset{1}{\tilde{B}}{}^a_i)+\overset{2}{l}{}^i_a(\overset{2}{\tilde{\pi}}{}^a_i-\overset{2}{\tilde{B}}{}^a_i)+\overset{1}{l}{}_i\overset{1}{\tilde{\pi}}{}^i+\overset{2}{l}{}_i\overset{2}{\tilde{\pi}}{}^i+\overset{1}{\tilde{m}}{}^a_i\overset{1}{\sigma}{}^i_a+\overset{2}{\tilde{m}}{}^a_i\overset{2}{\sigma}{}^i_a+\overset{1}{\tilde{s}}{}_i\overset{1}{\tilde{\psi}}{}^i+\overset{2}{\tilde{s}}{}_i\overset{2}{\tilde{\psi}}{}^i+\\ &+\overset{1}{n}{}^i_a\overset{1}{\tilde{\nu}}{}^a_i+\overset{2}{n}{}^i_a\overset{2}{\tilde{\nu}}{}^a_i+\overset{1}{\tilde{s}}{}^i_a\left[\overset{1}{\tilde{\psi}}{}^a_i-\frac{1}{2}\left(\tilde{\eta}^{abc}\overset{2}{\nabla}_b\overset{2}{\eta}{}^i_c-\overset{2}{\tilde{B}}{}^a_i\right)\right]+\overset{2}{\tilde{s}}{}^i_a\left[\overset{2}{\tilde{\psi}}{}^a_i-\frac{1}{2}\left(\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}^i_c-\overset{1}{\tilde{B}}{}^a_i\right)\right]\Big\}\,,\end{aligned}\tag{25}$$

where $\overset{1}{l}{}^i_a$, $\overset{2}{l}{}^i_a$, $\overset{1}{l}{}_i$, $\overset{2}{l}{}_i$, $\overset{1}{\tilde{m}}{}^a_i$, $\overset{2}{\tilde{m}}{}^a_i$, $\overset{1}{n}{}^i_a$, $\overset{2}{n}{}^i_a$, $\overset{1}{\tilde{s}}{}^i_a$, $\overset{2}{\tilde{s}}{}^i_a$, $\overset{1}{\tilde{s}}{}_i$, and $\overset{2}{\tilde{s}}{}_i$ are Lagrange multipliers. By imposing stability of the primary constraints under the evolution defined by the total

[12] Functional derivatives with respect to $\mathcal{L}_{\mathbf{t}}$ of the corresponding fields.

Hamiltonian H_T we get secondary constraints and equations involving the Lagrange multipliers. The secondary constraints are

$$\begin{cases} \overset{1}{\nabla}_a \overset{1}{\tilde{B}}{}^a_i - \epsilon_{ijk} \overset{1}{\tilde{\psi}}{}^a_j \overset{1}{\eta}_{ak} = 0 \\ \overset{2}{\nabla}_a \overset{2}{\tilde{B}}{}^a_i - \epsilon_{ijk} \overset{2}{\tilde{\psi}}{}^a_j \overset{2}{\eta}_{ak} = 0 \end{cases} \begin{cases} \overset{1}{\tilde{F}}{}^a_i + \overset{2}{\tilde{B}}{}^a_i - \tilde{\eta}^{abc} \overset{2}{\nabla}_b \overset{2}{\eta}_{ci} = 0 \\ \overset{2}{\tilde{F}}{}^a_i + \overset{1}{\tilde{B}}{}^a_i - \tilde{\eta}^{abc} \overset{1}{\nabla}_b \overset{1}{\eta}_{ci} = 0 \end{cases} \begin{cases} \overset{1}{\nabla}_a \left[\tilde{\eta}^{abc} \overset{2}{\nabla}_b \overset{2}{\eta}_{ci} - \overset{2}{\tilde{B}}{}^a_i \right] = 0 \\ \overset{2}{\nabla}_a \left[\tilde{\eta}^{abc} \overset{1}{\nabla}_b \overset{1}{\eta}_{ci} - \overset{1}{\tilde{B}}{}^a_i \right] = 0. \end{cases} \tag{26}$$

We also find a set of equations that the Lagrange multipliers must satisfy. In this case the stability of the secondary constraints does not give any new ones but only additional equations for the Lagrange multipliers. These equations must be treated with care. In many cases it is possible to show that they can be solved for any configuration of the fields in which case it is not necessary to solve them explicitly in order to get the final form of the constraints. However, it may happen[13] that some restrictions on the fields and momenta must be imposed in order to guarantee their solvability. These conditions are new secondary constraints that must be put in equal footing with the remaining ones, in particular we must impose their stability that may lead to new equations and constraints. In the present case a long and tedious computation shows that no new constraints must be enforced to ensure the solvability of the equations for the Lagrange multipliers.

By using the explicit expression for the Lagrange multipliers and the fact that, once they are plugged in H_T we can express it as the sum of a first class Hamiltonian and a linear combination of the primary first class constraints with arbitrary functions of time as coefficients we find out that $\overset{1}{A}{}^i_0$, $\overset{2}{A}{}^i_0$, $\overset{1}{\eta}{}^i_0$, $\overset{2}{\eta}{}^i_0$, $\overset{1}{\beta}{}^i_0$, $\overset{2}{\beta}{}^i_0$ are arbitrary functions and the following first class constraints

$$\begin{aligned} &\overset{1}{\tilde{F}}{}^a_i + \overset{2}{\tilde{B}}{}^a_i - \tilde{\eta}^{abc} \overset{2}{\nabla}_b \overset{2}{\eta}_{ci} = 0 \\ &\overset{2}{\tilde{F}}{}^a_i + \overset{1}{\tilde{B}}{}^a_i - \tilde{\eta}^{abc} \overset{1}{\nabla}_b \overset{1}{\eta}_{ci} = 0 \\ &2\overset{1}{\nabla}_a \overset{1}{\tilde{B}}{}^a_i + \epsilon_{ijk} \left(\overset{2}{\tilde{B}}{}^a_j - \tilde{\eta}^{abc} \overset{2}{\nabla}_b \overset{2}{\eta}_{cj} \right) \overset{1}{\eta}_{ak} = 0 \\ &2\overset{2}{\nabla}_a \overset{2}{\tilde{B}}{}^a_i + \epsilon_{ijk} \left(\overset{1}{\tilde{B}}{}^a_j - \tilde{\eta}^{abc} \overset{1}{\nabla}_b \overset{1}{\eta}_{cj} \right) \overset{2}{\eta}_{ak} = 0 \end{aligned} \tag{27}$$

[13]As, for example, if one considers the action (7).

where the last two equations can be substituted by

$$
\begin{aligned}
\stackrel{1}{\nabla}_a \overset{2}{\tilde{F}}{}^a_i &= 0 \\
\stackrel{2}{\nabla}_a \overset{1}{\tilde{F}}{}^a_i &= 0
\end{aligned}
\tag{28}
$$

by using the first pair. We also find the following second class constraints

$$
\begin{aligned}
\overset{1}{\tilde{\pi}}{}^a_i - \overset{1}{\tilde{B}}{}^a_i &= 0 \\
\overset{2}{\tilde{\pi}}{}^a_i - \overset{2}{\tilde{B}}{}^a_i &= 0 \\
\overset{1}{\sigma}{}^i_a &= 0 \\
\overset{2}{\sigma}{}^i_a &= 0 \\
2\overset{1}{\tilde{\psi}}{}^a_i - \left(\tilde{\eta}^{abc} \stackrel{2}{\nabla}_b \overset{2}{\eta}{}^i_c - \overset{2}{\tilde{B}}{}^a_i\right) &= 0 \\
2\overset{2}{\tilde{\psi}}{}^a_i - \left(\tilde{\eta}^{abc} \stackrel{1}{\nabla}_b \overset{1}{\eta}{}^i_c - \overset{1}{\tilde{B}}{}^a_i\right) &= 0\,.
\end{aligned}
\tag{29}
$$

The symplectic structure for the initial set of canonical variables (after the elimination of the arbitrary objects $\overset{1}{A}{}^i_0$, $\overset{2}{A}{}^i_0$, $\overset{1}{\eta}{}^i_0$, $\overset{2}{\eta}{}^i_0$, $\overset{1}{\beta}{}^i_0$, $\overset{2}{\beta}{}^i_0$) is

$$
\begin{aligned}
\Omega = -2\int_\Sigma d^3x \Big[& \mathbb{d}\overset{1}{A}{}^i_a(x) \wedge \mathbb{d}\overset{1}{\tilde{\pi}}{}^a_i(x) + \mathbb{d}\overset{1}{\tilde{B}}{}^a_i(x) \wedge \mathbb{d}\overset{1}{\sigma}{}^i_a(x) + \mathbb{d}\overset{1}{\eta}{}^i_a(x) \wedge \mathbb{d}\overset{1}{\tilde{\psi}}{}^a_i(x) + \\
& + \mathbb{d}\overset{2}{A}{}^i_a(x) \wedge \mathbb{d}\overset{2}{\tilde{\pi}}{}^a_i(x) + \mathbb{d}\overset{2}{\tilde{B}}{}^a_i(x) \wedge \mathbb{d}\overset{2}{\sigma}{}^i_a(x) + \mathbb{d}\overset{2}{\eta}{}^i_a(x) \wedge \mathbb{d}\overset{2}{\tilde{\psi}}{}^a_i(x) \Big]\,.
\end{aligned}
$$

Now we must pull it back onto the hypersurface in phase space defined by the second class constraints. As a first step we use (29) to remove $\overset{1}{\sigma}{}^i_a$, $\overset{2}{\sigma}{}^i_a$, $\overset{1}{\tilde{B}}{}^a_i$, and $\overset{2}{\tilde{B}}{}^a_i$. At this stage we are left with the first class constraints

$$
\begin{aligned}
\overset{1}{\tilde{F}}{}^a_i + \overset{2}{\tilde{\pi}}{}^a_i - \tilde{\eta}^{abc} \stackrel{2}{\nabla}_b \overset{2}{\eta}{}_{ci} &= 0 \\
\overset{2}{\tilde{F}}{}^a_i + \overset{1}{\tilde{\pi}}{}^a_i - \tilde{\eta}^{abc} \stackrel{1}{\nabla}_b \overset{1}{\eta}{}_{ci} &= 0 \\
\stackrel{1}{\nabla}_a \overset{2}{\tilde{F}}{}^a_i &= 0 \\
\stackrel{2}{\nabla}_a \overset{1}{\tilde{F}}{}^a_i &= 0\,,
\end{aligned}
\tag{30}
$$

the second class constraints

$$
\begin{aligned}
2\overset{1}{\tilde{\psi}}{}^a_i - \left(\tilde{\eta}^{abc} \stackrel{2}{\nabla}_b \overset{2}{\eta}{}^i_c - \overset{2}{\tilde{\pi}}{}^a_i\right) &= 0 \\
2\overset{2}{\tilde{\psi}}{}^a_i - \left(\tilde{\eta}^{abc} \stackrel{1}{\nabla}_b \overset{1}{\eta}{}^i_c - \overset{1}{\tilde{\pi}}{}^a_i\right) &= 0\,,
\end{aligned}
\tag{31}
$$

and the symplectic structure

$$\Omega=-2\int_\Sigma d^3x\left[\mathbb{d}\,\overset{1}{A}{}^i_a(x)\wedge\mathbb{d}\,\overset{1}{\tilde{\pi}}{}^a_i(x)+\mathbb{d}\overset{1}{\eta}{}^i_a(x)\wedge\mathbb{d}\,\overset{1}{\tilde{\psi}}{}^a_i(x)+\right.$$
$$\left.+\mathbb{d}\,\overset{2}{A}{}^i_a(x)\wedge\mathbb{d}\,\overset{2}{\tilde{\pi}}{}^a_i(x)+\mathbb{d}\overset{2}{\eta}{}^i_a(x)\wedge\mathbb{d}\,\overset{2}{\tilde{\psi}}{}^a_i(x)\right].$$

We have now two possible choices:

i) Remove $\overset{1}{\tilde{\psi}}{}^a_i$ and $\overset{2}{\tilde{\psi}}{}^a_i$ by using (31) to get a phase space coordinatized by $\overset{1}{A}{}^i_a$, $\overset{2}{A}{}^i_a$, $\overset{1}{\tilde{\pi}}{}^i_a$, $\overset{2}{\tilde{\pi}}{}^i_a$, $\overset{1}{\eta}{}^i_a$, and $\overset{2}{\eta}{}^i_a$, the first class constraints (30) and the symplectic structure

$$\Omega=-\int_\Sigma d^3x\left[2\mathbb{d}\,\overset{1}{A}{}^i_a(x)\wedge\mathbb{d}\,\overset{1}{\tilde{\pi}}{}^a_i(x)+\mathbb{d}\overset{1}{\eta}{}^i_a(x)\wedge\mathbb{d}\left(\tilde{\eta}^{abc}\overset{2}{\nabla}_b\overset{2}{\eta}{}_{ci}(x)-\overset{2}{\tilde{\pi}}{}^a_i(x)\right)+\right.$$
$$\left.+2\mathbb{d}\,\overset{2}{A}{}^i_a(x)\wedge\mathbb{d}\,\overset{2}{\tilde{\pi}}{}^a_i(x)+\mathbb{d}\overset{2}{\eta}{}^i_a(x)\wedge\mathbb{d}\left(\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}_{ci}(x)-\overset{1}{\tilde{\pi}}{}^a_i(x)\right)\right].$$

As we have 54 canonical variables and 24 first class constraints we have three physical degrees of freedom per space point. The Dirac brackets of the phase space variables can be obtained in closed form by inverting the previous expression for Ω in matrix form.

ii) Remove $\overset{1}{\tilde{\pi}}{}^a_i$ and $\overset{2}{\tilde{\pi}}{}^a_i$ to get a phase space coordinatized by $\overset{1}{A}{}^i_a$, $\overset{2}{A}{}^i_a$, $\overset{1}{\tilde{\psi}}{}^i_a$, $\overset{2}{\tilde{\psi}}{}^i_a$, $\overset{1}{\eta}{}^i_a$, and $\overset{2}{\eta}{}^i_a$. The first class constraints are given now by

$$\begin{aligned}
&\overset{1}{\tilde{F}}{}^a_i-2\overset{1}{\tilde{\psi}}{}^a_i=0\\
&\overset{2}{\tilde{F}}{}^a_i-2\overset{2}{\tilde{\psi}}{}^a_i=0\\
&\overset{1}{\nabla}_a\overset{2}{\tilde{F}}{}^a_i=0\\
&\overset{2}{\nabla}_a\overset{1}{\tilde{F}}{}^a_i=0
\end{aligned}$$

and the symplectic structure is

$$\Omega=-2\int_\Sigma d^3x\left[\mathbb{d}\,\overset{1}{A}{}^i_a(x)\wedge\mathbb{d}\left(\tilde{\eta}^{abc}\overset{1}{\nabla}_b\overset{1}{\eta}{}_{ci}(x)-2\overset{2}{\tilde{\psi}}{}^a_i(x)\right)+\mathbb{d}\overset{1}{\eta}{}^i_a(x)\wedge\mathbb{d}\,\overset{1}{\tilde{\psi}}{}^a_i(x)+\right.$$
$$\left.+\mathbb{d}\,\overset{2}{A}{}^i_a(x)\wedge\mathbb{d}\left(\tilde{\eta}^{abc}\overset{2}{\nabla}_b\overset{2}{\eta}{}_{ci}(x)-2\overset{1}{\tilde{\psi}}{}^a_i(x)\right)+\mathbb{d}\overset{2}{\eta}{}^i_a(x)\wedge\mathbb{d}\,\overset{2}{\tilde{\psi}}{}^a_i(x)\right].$$

As before we have 54 canonical variables and 24 first class constraints per space point. In both cases we can recover in a straightforward way the formulation of [8] by using the consistent gauge fixing conditions

$$\overset{1}{\eta}{}^i_a=0$$

$$\overset{2}{\eta}{}^i_a = 0.$$

If instead of using the action (9) we take (7) the procedure is considerably more involved due to the appearance of several layers of secondary constraints that come up as consistency conditions for the solvability of the Lagrange multipliers equations that are avoided here by using the Stückelberg procedure. Anyway, as argued in the main text, the formulation is strictly equivalent to the other ones presented in the paper.

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